ISRAEL JOURNAL OF MATHEMATICS **132** (2002), 175–187

AUTOMORPHIC FORMS AND THE DISTRIBUTION OF POINTS ON ODD-DIMENSIONAL SPHERES

BY

L. CLOZEL

Université de Paris-Sud, UMR 8628 du CNRS, Mathématique, bât. 425
91405 Orsay Cedex, France
e-mail: clozel@math.u-psud.fr

Sie wissen, wir brauchen immer einen Vorwand für unsere Stadtväter, um unseren Gelehrten etwas zukommen zu lassen.

Brecht, Leben des Galilei

ABSTRACT

We extend to the case of *odd-dimensional* spheres a theorem of Lubotzky, Phillips and Sarnak giving optimally equidistributed sets of points. The proof relies on the theory of automorphic forms and higher-dimensional Shimura varieties.

Introduction

In 1986 Lubotzky, Phillips and Sarnak [9] applied the theory of automorphic forms to the problem of distributing sequences of points on the sphere S^2. They considered families of increasing size of elements of $SO(3)$, say

$$\{\gamma_1, \gamma_2, \ldots, \gamma_N, \gamma_1^{-1}, \ldots \gamma_N^{-1}\} = S,$$

and considered the operator

$$T_S = \sum_{\gamma \in S} \gamma$$

operating on the space $L_0^2(S^2)$ of L^2 functions of zero mean. For suitable choices of the $\{\gamma_i\}$ they showed that the operator norm of T_S is bounded by $\frac{1}{N}\sqrt{2N-1}$;

Received December 21, 2000

they also showed that this bound is optimal. A bound on the operator norm implies, for any $x \in S^2$, a bound on

$$\left|\frac{1}{2N}T_S f(x) - \int_{S^2} f\right|$$

for any suitably regular function f on S^2. Thus the T_S yield a finite approximation of the integral, with a good error term.

The purpose of this paper is to extend some of these results to **odd-dimensional** spheres. Thus let $n \geq 2$ be an integer and S^{2n-1} the sphere $\|x\| = 1$ in $\mathbb{R}^{2n}$ with the standard Euclidian metric. A Hecke operator is an operator on $L^2(S^{2n-1})$ of the form

$$T = T_S = \sum_{\gamma \in S} \gamma$$

where $\gamma \in SO(2n)$. We denote by $d = \deg(T)$ the degree of T; thus $d = |S|$. As in [9] we set

$$\delta(T) = \frac{1}{\deg T}\|T \mid L_0^2(S^{2n-1})\|$$

where the norm is the L^2-operator norm. Note that our operators are not symmetric.

THEOREM 1: *There exists a positive integer $h \geq 1$ with the following property.*

For each prime $p \equiv 1$ [4], $p \geq 13$ there exists a Hecke operator T_p of degree $d = h(p^{n-1} + \cdots + p + 1)$ such that

$$\delta(T_p) \leq h\, n\, p^{(1-n)/2}.$$

COROLLARY: *For any given $\varepsilon > 0$,*

$$\delta(T_p) \leq h^{3/2} n(1+\varepsilon) d(T_p)^{-1/2} \quad (p \to +\infty).$$

The constant h is essentially a class number, which could be determined with the requisite amount of (mechanical) computation. For large n it is not possible to obtain $h = 1$, as in [9]. For heuristic purposes assume, however, that this is attained. We can then ask whether the bound

$$\delta(T_p) \leq n\; p^{(1-n)/2} \qquad (??)$$

is optimal. One can show at least that the order of growth of δ, with respect to the degree, is indeed optimal.

We identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ endowed with the standard Hermian metric. We will constuct our operators T_p from elements $\gamma \in U(n)$ acting naturally on the sphere. They then also act on the quotient $\mathbb{P}^{n-1}(\mathbb{C})$ of S^{2n-1}. An argument of [9] applies in this situation to yield:

THEOREM 2: *It T is a symmetric Hecke operator (associated to elements of $U(n)$) acting on $L_0^2(\mathbb{P}^{n-1}(\mathbb{C}))$, of degree $d = 2N$, $\delta(T) \geq \sqrt{2N-1}/N$.*

This has been proved by Pisier [10] for Hecke operators acting on the space of functions of mean value 0 on the sphere. We include the proof, however, because it is quite different and gives a natural extension of the methods of [9]. More general results have been announced by Y. Shalom [12, §4.4].

In particular, this lower bound is *a fortiori* true for the action on $L_0^2(S^{2n-1})$. It is likely that Theorem 2 could be proved, with no restriction on parity, for $L_0^2(S^n)$. This was suggested by the authors of [9]: see p. 164 of that paper.

Consider an operator T_p as in Theorem 1, and set $T = T_p + T_p^*$. Then, $\delta(T) \leq 2\delta(T_p)$. (Here both operators are considered in $L^2(S^{2n-1})$.) Thus Theorem 1 and Theorem 2 yield an upper bound and a lower bound on $\delta(T)$, which are easily seen to be compatible. We think, however, that in higher dimension the lower bound should be more like $n\ p^{(1-n)/2}$ (see (??) above), which of course has no evident expression in terms of the degree. If we may assume $h = 1$, this can be proved using the methods of Serre [11] which allow one to study the repartition of the eigenvalues of T_p in spaces of spherical harmonics of high degree. We defer this to an eventual application.

We have not spelled out all the consequences of Theorem 1; many would follow simply by the methods of [9]. We only state the obvious consequence for finite integration on S^{2n-1}. Let Δ denote the invariant Laplace operator on the sphere.

THEOREM 3: *Assume f is a function on S^{2n-1} such that $\Delta^n f$ is square-integrable (so f is continuous). Then for any $x \in S^{2n-1}$*

$$\left| \frac{1}{\deg(T_p)} \sum_{\gamma} f(\gamma x) - \int_{S^{2n-1}} f \right| \leq h\, n\, p^{(1-n)/2} \{ C_1 \|f\|_2 + C_2 \|\Delta^n f\|_2 \}.$$

The constants C_1 and C_2 are independent of f and obtained as follows. Let φ, ψ be two functions on the sphere such that $\Delta^n \varphi = \delta + \psi$, δ being the Dirac measure at some point x_0 on the sphere. We assume ψ to be C^∞ and φ continuous. Then $C_1 = \|\psi\|_2$ and $C_2 = \|\varphi\|_2$. The proof (in a more general context) can be found in [4, §8]. We repeat it.

Let $G = U(n)$, and let $K \cong U(1) \times U(n-1)$ be the stabilizer of the point $x \in S^{2n-1}$. The Laplacian Δ on the sphere can be seen as an element of $U(\mathfrak{g})^K$, a commutative algebra since (G, K) is a symmetric pair. We in turn consider Δ as a distribution on G with support at the origin. Let δ be the Dirac measure at x. It is well-known that there exist functions φ, ψ on $S^{2n-1} = G/K$ such that

$$\varphi * \Delta^n = \delta + \psi$$

with ψ C^∞ and φ continuous. Averaging, we may assume φ, ψ invariant by K. Then this can be rewritten as

$$\Delta^n * \varphi = \delta + \psi.$$

Now the argument of [4, §8] yields, with $\widetilde{T} = \frac{1}{\deg(T_p)} T_p$,

$$\widetilde{T}f - f_0 = [\widetilde{T}(f * \Delta^n) - (f * \Delta^n)_0] * \varphi - [\widetilde{T}(f - f_0)] * \psi,$$

g_0 denoting for any function g on S^{2n-1} the constant function equal to $\int_{S^{2n-1}} g$. Since $|g * h|(x) \leq \|g\|_2 \|h\|_2$ for two functions on G, the result follows.

The proof of Lubotzky, Phillips and Sarnak relied on the Ramanujan conjectures proved by Deligne. Our proof relies on higher-dimensional analogues of the Ramanujan conjecture, and implicitly on deep results concerning the Hasse–Weil zeta functions of Shimura varieties and obtained in the last decade. As well as Deligne's general solution of the Weil conjectures [6], we use Kottwitz's results on special Shimura varieties [8] and the solution given by us of a conjecture of Rapoport.

Finally, we can obtain an **explicit** set of primes (here $p \geq 13$) only by using the solution by Harris and Taylor of the local Langlands conjecture!

We note that we do not know how to extend these results to even-dimensional spheres. Here, however, weaker estimates are available as a consequence of [3] and are contained in a paper by H. Oh.*

1. Hecke operators on certain unitary groups

1.1. We view S^{2n-1} as a quotient of $G = U(n)$. A Hecke operator is an operator on $L^2(G)$ of the form $T = \sum_{\gamma \in S} \gamma$, $S \subset G$. We use the notations of the Introduction. Theorem 1 follows from

THEOREM 1.1: *For $p \equiv 1$ [4], $p \geq 13$ there exists a Hecke operator T_p of degree $d = h(p^{n-1} + \cdots + 1)$ such that*

$$\delta(T_p) = \frac{1}{d} \|T_p \mid L^2_{00}(G)\| \leq h\, n\, p^{(1-n)/2}.$$

* H. Oh, *Distributing points on S^N $(N \geq 4)$, après* Lubotzky, Phillips and Sarnak.

We will in fact construct an infinite number of such operators. Here $L^2_{00}(G)$ is the space of functions on G orthogonal to the Abelian characters.

In this paragraph we have to recall the formalities of "automorphic forms on compact groups". Since $U(n)$ plays no particular role, we consider an arbitrary reductive group G over $\mathbb{Q}$ such that $G(\mathbb{R})$ is compact. (Thus $U(n)$ will be $G(\mathbb{R})$.) Write $G(\mathbb{A}) = G(\mathbb{R}) \times G(\mathbb{A}_f)$. Let K be a compact open subgroup of $G(\mathbb{A}_f)$. Then we can choose a finite set of elements $(g_i)_{i\in I}$ of $G(\mathbb{A}_f)$ such that

$$G(\mathbb{A}) = \coprod_{i\in I} G(\mathbb{Q})G(\mathbb{R})g_i K. \tag{1.1}$$

If, moreover, $G(\mathbb{R}) \cdot g\, K\, g^{-1} \cap G(\mathbb{Q}) = \{1\}$ for any $g \in G(\mathbb{A}_f)$, (1.1) yields

$$X_K := G(\mathbb{Q})\backslash G(\mathbb{A})/K \cong \coprod_{i\in I} G(\mathbb{R}), \tag{1.2}$$

the embedding of each factor $G(\mathbb{R})$ being given by $g_\infty \mapsto g_\infty g_i$. We will make this assumption. Let $h = \#I$.

In particular, the space of smooth functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$ invariant by K is identified with $\bigoplus_i C^\infty(G(\mathbb{R}))$; ditto with L^2 functions.

Suppose $x \in G(\mathbb{A}_f)$. Our operator T will be given by the action on the right of the double coset $K\,x\,K$ on $L^2(X_K)$. We identify $L^2(X_K)$ with L^2-functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$, invariant by K on the right. Set

$$K\,x\,K = \coprod K\,x\,\kappa_j, \quad h_j = x\,\kappa_j \in G(\mathbb{A}_f), \tag{1.3}$$

a finite sum, where $\kappa_j (j \in J)$ ranges over $x^{-1}K\,x \cap K\backslash K$. Then

$$(f|T)(g) = \sum f(g\, h_j). \tag{1.4}$$

By (1.2) $L^2(X_K) \cong \bigoplus_{i\in I} L^2(G(\mathbb{R}))$. For $f \in L^2(X_K)$ write $f = (f_i)$. Then $f|T = \varphi = (\varphi_\alpha)_{\alpha\in I}$. For each α, $\varphi_\alpha(g_\infty) = (f|T)(g_\infty g_\alpha) = \sum_j f(g_\infty g_\alpha h_j)$.

Note that (1.1) implies

$$G(\mathbb{A}_f) = \coprod_{i\in I} G(\mathbb{Q})g_i K. \tag{1.5}$$

Thus we can write, for each pair (α, j),

$$g_\alpha h_j = \gamma(\alpha, j) g_{i(\alpha,j)} \kappa(\alpha, j) \tag{1.6}$$

with $\gamma(\alpha, j) \in G(\mathbb{Q})$ and $\kappa(\alpha, j) \in K$.

For clarity write $(g_\infty; g_f)$ for an element of $G(\mathbb{A}) = G(\mathbb{R}) \times G(\mathbb{A}_f)$. Then $\gamma(g_\infty; g_f)$ means $\gamma \in G(\mathbb{Q})$ acts diagonally, while γg_∞, γg_f denote the separate actions. Now

$$\varphi_\alpha(g_\infty) = \sum_j f((g_\infty, g_\alpha h_j)) = \sum_j f((g_\infty; \gamma(\alpha, j) g_{i(\alpha,j)})$$

(since f is K-invariant)

$$= \sum_j f((\gamma(\alpha,j)^{-1} g_\infty; g_{i(\alpha,j)})$$

(since f is $G(\mathbb{Q})$-invariant)

$$= \sum_j f_{i(\alpha,j)}(\gamma(\alpha,j)^{-1} g_\infty).$$

We record this as

Lemma 1.1: $(f|T)_\alpha(g_\infty) = \sum_j f_{i(\alpha,j)}(\gamma(\alpha,j)^{-1} g_\infty)$.

Now consider $L^2(G(\mathbb{R}))$ embedded diagonally in $L^2(X_K)$ by $f \mapsto (f, \dots, f)$; denote this embedding by J; let S: $L^2(X_K) \to L^2(G(\mathbb{R}))$ be the sum, $(f_i) \mapsto \sum f_i$. Then the operator $S\,T\,J$ sends f to

$$S\,T\,J(f)(g_\infty) = \sum_\alpha \sum_j f(\gamma(\alpha,j)^{-1} g_\infty).$$

Thus it is a Hecke operator in the sense of the Introduction. Since the norms of J and of S are both equal to $\sqrt{h}$, we have for the L^2-operator norms:

Lemma 1.2: $\|S\,T\,J\| \le h\|T\|$.

Denote by $L^2_{00}(X_K)$ the subspace of functions orthogonal to all 1-dimensional characters of $G(\mathbb{R})^h$. Since T commutes with the diagonal action of $G(\mathbb{R})$ on the right, T preserves L^2_{00}; J sends $L^2_{00}(G(\mathbb{R}))$ (defined analogously) to $L^2_{00}(X_K)$ and conversely for S. Thus:

Lemma 1.3: $\|S\,T\,J \mid L^2_{00}(G(\mathbb{R}))\| \le h\|T \mid L^2_{00}(G(\mathbb{R}))\|$.

1.2. Now $G(\mathbb{R}) = U(n)$ and we must construct the group $G/\mathbb{Q}$. We use the groups constructed in [8]. Let E be the field $\mathbb{Q}(\sqrt{-1})$ and let D be a division algebra over E satisfying the following properties:
(1.7i) D admits an involution of the second kind $*$ relative to $E/\mathbb{Q}$.

(1.7ii) At the two primes $\mathbf{5}'$, $\mathbf{5}''$ of E dividing 5, D has invariant $1/n$, $-1/n$; it is thus a division algebra over these local fields.
(1.7iii) The involution $(*)$ is positive on $D \otimes \mathbb{C}$.

Let G be the unitary group over $\mathbb{Q}$ defined by D:

$$G(\mathbb{Q}) = \{d \in D : dd^* = 1\}.$$

Then $G(\mathbb{R}) \cong U(n)$ by (1.7iii). We further assume:
(1.7iv) G is quasi-split and unramified at all primes $p \neq 2, 3$ and 5.

That G can be so chosen (for a suitable choice of D and $*$) is proved in [1, §2].

We now choose the group $K = \prod_p K_p \subset G(\mathbb{A}_f)$. The group $G(\mathbb{Q}_5)$ is isomorphic to $D^\times(E_{\mathbf{5}'})$, a division algebra over $\mathbb{Q}_5$. Let K_5 be its maximal compact subgroup. For $p > 5$, $G(\mathbb{Q}_p)$ is unramified — a unitary group or $\mathrm{GL}(n)$ — and has a natural hyperspecial subgroup K_p. Finally, we choose K_2 or K_3 small enough that the assumptions in §1.1 are met. Of course we have no control on the resulting class number h.

The operator T_p of Theorem 1.1 will be $S\,T\,J$, where T is a Hecke operator in the Hecke algebra of $(G(\mathbb{Q}_p), K_p)$ for $p \equiv 1$ [4]. Then $G(\mathbb{Q}_p) = \mathrm{GL}(n, \mathbb{Q}_p)$, $K_p = \mathrm{GL}(n, \mathbb{Z}_p)$ and

$$T = K_p \begin{pmatrix} p & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} K_p.$$

The space $\mathcal{H}_K$ is $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K) = L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))^K$. The space

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})) := \mathcal{L}_G$$

decomposes discretely as a sum of automorphic representations of $G(\mathbb{A})$, of the form $\pi = \pi_\infty \otimes \bigotimes_p \pi_p$.

Recall [1] that π_p determines its Hecke matrix, a diagonal matrix $t_{\pi,p}$ in $\mathrm{GL}(n, \mathbb{C})$. Then, with the implicit normalizations we chose, the associated operator T acts by $p^{(n-1)/2}(t_1 + \cdots + t_n)$ where (t_i) are the eigenvalues of $t_{\pi,p}$. We now have:

Proposition 1.4: *If π_∞ is not an Abelian character, $|t_i| = 1$.*

Proof: We show that the representation π of $G(\mathbb{A})$ lifts to an automorphic representation π_E of $G(\mathbb{A}_E) = D^\times(\mathbb{A}_E)$ such that
(1.8) For $p > 5$ and $\mathfrak{p}|p$, $\pi_{E,\mathfrak{p}}$ is obtained by unramified base change from π_p.
(1.9) $\pi_{E,\mathbf{5}'}$ and $\pi_{E,\mathbf{5}''}$ are Abelian characters.

(1.10) The infinitesimal characters of the Archimedian components π_∞ and $\pi_{E,\infty}$ are associated by base change (cf. [2]).

The argument is essentially contained in Clozel–Labesse [2: Théorème A.5.2], whose notations we keep. Let $S = \{2,3,5\}$, and choose small compact-open subgroups K_2', K_3' of K_2, K_3. Then if $K' = K_2'K_3'K_5\prod_{p>5} K_p$, $\pi^{K'}$ occurs in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K')$. We take a function f on $G(\mathbb{A})$ of the form $f_\infty \otimes \bigotimes_p f_p$, where f_p is the characteristic function of K_p', and f_∞ is the normalized character of π_∞. If r denotes the representation for $p \leq 5$, of $G(\mathbb{A})$ on $\mathcal{L}_G$, and $f^S = \bigotimes_{p\notin S} f_p$ is unramified,

$$\begin{aligned}\operatorname{tr} r(f) =\{ &\sum_{\substack{\rho: \\ \chi_\rho^S=\chi_\pi^S}} \operatorname{tr}\rho_\infty(f_\infty)\operatorname{tr}\rho_S(f_S)\}\chi_\rho(f^S) \\ &+ \sum_{\substack{\rho: \\ \chi_\rho^S\neq\chi_\pi^S}} \operatorname{tr}\rho_\infty(f_\infty)\operatorname{tr}\rho_S(f_S)\chi_\rho(f^S),\end{aligned}$$

the sums being finite even for varying f^S. The coefficient $\{\cdots\}$ is strictly positive. For K_2', K_3' small enough, the functions f_2 and f_3 are associated to functions φ_2, φ_3 on $G(E\otimes\mathbb{Q}_2)$, $G(E\otimes\mathbb{Q}_3)$. The same is true for the function f_5 since 5 splits. The proof now follows as in [2: Théorème A.5.2] from the base change identity [2: Théorème A.3.1]; by the correspondence of Hecke functions (1.8) is satisfied, and (1.9) also is satisfied by split base change at 5.

Now p splits in E, so π_p determines a representation of $D^\times(E\otimes\mathbb{Q}_p) = \mathrm{GL}(n,\mathbb{Q}_p)\times\mathrm{GL}(n,\mathbb{Q}_p)$, isomorphic to $\pi_p\times\pi_p$ (up to identifications). We must show the "purity" condition $|t_i| = 1$. For this, note that by the Jacquet–Langlands correspondence — due in this case to Vignéras — π_E is associated to an automorphic (discrete) representation π_E' of $\mathrm{GL}(n,\mathbb{A}_E)$; since π_E is an Abelian character at $\mathbf{5}'$ or $\mathbf{5}''$, π_E' is isomorphic to an Abelian twist of the Steinberg representation at those primes, and therefore **cuspidal**, or is an Abelian character [13]. The last possibility is ruled out because $\pi_{E,\infty}' \cong \pi_{E,\infty}$, and the infinitesimal character of $\pi_{E,\infty}$ comes from that of π_∞ by base change: but π_∞ is not an Abelian character. Harris and Taylor then show that there is a Galois representation — pure of weight 0 — whose Frobenius eigenvalues are associated in the usual fashion to the Hecke matrices [7, Theorem C]. This implies that $|t_i| = 1$. (If we were content with almost all primes we could use the results in [1, 2].)

We note that Theorem 1 follows from Theorem 1.1: since S^{2n-1} is the quotient of $U(n)$ by the subgroup $U(1)\times U(n-1)$ stabilizing the first basis vector, $L^2(S^{2n-1})$ is a subspace of $L^2(U(n))$. Then $L_0^2(S^{2n-1}) = L_{00}^2(U(n))\cap L^2(S^{2n-1})$.

Use Lemmas 1.2 and 1.3. Finally, we remark that obviously other sets of primes, associated to different quadratic fields, would be obtained analogously (but with, in each case, a different h).

2. Proof of Theorem 2

The proof adapts the methods of [9] to our case. We follow the exposition of Colin de Verdière [5]. However, a more delicate representation-theoretic argument is needed in addition to [9]: this is due to the fact that our Lie group (here $PU(n)$, see below), unlike $SO(3)$, has singular elements different from 1.

Since we are working in $L^2(\mathbb{P}^{n-1}(\mathbb{C}))$, the unitary group $U(n)$ acts in fact via the projective unitary group $PU(n)$, which we denote by G. Thus $S = \{\gamma_1, \gamma_2, \dots \gamma_N, \gamma_1^{-1}, \dots, \gamma_N^{-1}\}$ can be taken to be a symmetric set in G; let T_S be the associated Hecke operator. We have

$$\mathbb{P}^{n-1}(\mathbb{C}) = G/H \quad \text{where } H = P(U(1) \times U(n-1)).$$

Let $T \subset H \subset G$ be the (diagonal) maximal torus. An irreducible representation of G is then determined by its highest weight. This is a character of $\widetilde{T} = U(1)^n \subset U(n)$, given by $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \dots \geq \lambda_n$; moreover $\Sigma\lambda_i = 0$.

We are interested in representations that occur in $L^2(G/H)$. Since (G, H) is a symmetric pair they occur with multiplicity one; by the theorem of Cartan–Helgason [14, Theorem 3.3.1.1] these are the representations such that $\lambda = (\lambda_1, \mu, \mu, \dots \mu)$. Thus $\lambda = ((n-1)r, -r, -r, \dots, -r)$ for $r \geq 0$. We will sometimes write $\lambda = \lambda_r$ ($r \in \mathbb{N}$). From now on λ will always denote such a weight.

Let $V_r \subset L^2(G/H)$ be the associated space, and let μ_r denote the atomic measure on $\mathbb{R}$ with support in $[-2N, 2N]$ — $2N$ being the operator norm of T_S — given by

$$\mu_r = \frac{1}{\dim V_r}\Big(\sum_{j=1}^{\dim V_r} \delta_{\lambda_j}\Big), \tag{2.1}$$

where $\{\lambda_j\}$ are the eigenvalues of T_S in V_r and δ_{λ_j} is the Dirac measure at λ_j. Theorem 2 follows from:

THEOREM 2.1: *For $r \to \infty$, $\mu_r \to \mu$, the spectral measure of the group Γ generated by S in G with respect to the generators S.*

We refer to [9] or [5] for the notion of spectral measure, and for the fact that Theorem 2.1 implies Theorem 2 (by a theorem of Kesten). Recall that μ, a measure on $[-2N, 2N]$, is determined by its moments: for $s \in \mathbb{N}$,

$$\int t^s d\mu = m_s = \#\{\gamma \in W_s : \gamma = 1 \text{ in } G\}. \tag{2.2}$$

Here, W_s is the set of words of length s in the generators $(\gamma_i, \gamma_i^{-1})$.

In order to prove Theorem 2.1, it is then sufficient to estimate $\int t^s d\mu_r$ for fixed s and $r \to +\infty$. Now

$$\int t^s d\mu_r = \frac{1}{\dim V_r} \Sigma \lambda_j^s = \frac{1}{\dim V_r} \operatorname{tr}(T_S^s | V_r)$$

and this is equal to

$$\frac{1}{\dim V_r} \sum_{\gamma \in W_s} \operatorname{tr}(\gamma | V_r). \tag{2.3}$$

If $\gamma = 1$ in G, $\frac{1}{\dim V_r} \operatorname{tr}(\gamma | V_r) = 1$. Since W_s is fixed it suffices to prove:

LEMMA 2.3: *If $g \in G$ and $g \neq 1$,*

$$\frac{1}{\dim V_r} \operatorname{tr}(g | V_r) \to 0 \quad (r \to +\infty).$$

Proof: Let $x \in U(n)$ be a lift of g for the natural map $U(n) \to G$; then x is not central. If x is regular the lemma follows immediately from Weyl's character formula (cf. [9]). Assume x singular. Its centralizer in $U(n)$ is then equal to $M = U(n_1) \times \cdots \times U(n_t)$ and different from $U(n) : t \geq 2$.

We may see V_r as a representation of $U(n) \supset M$. The character of its restriction to M is then given by Kostant's formula:

$$\operatorname{ch}_M(V_r) = \frac{\sum_w \varepsilon(w) \operatorname{ch}_M(V^M_{w(\lambda+\rho)-\rho})}{\sum_w \varepsilon(w) \operatorname{ch}_M(V_{w\rho-\rho})}. \tag{2.4}$$

Here w ranges over elements of the set $\Omega^M \subset \Omega$, Ω being the Weyl group of (G,T) and Ω^M the subset such that $w^{-1}\Delta_M^+ \subset \Delta_G^+$ — here $\Delta_G = \Delta_G^- \cup \Delta_G^+$ is the root system of (G,T) for some choice of Borel subgroup and $\Delta_M^+ = \Delta_G^+ \cap \Delta_M$, with obvious notation. We have indexed irreducible modules V^M for M by their highest weight. See [14, Theorem 2.4.2.2] for the other (standard) notations.

A priori this is a formal identity in the Grothendieck group of representations of M, but it can in fact be evaluated on an element x in the center of M. For this it suffices to check that the denominator is non-zero. Using Weyl's formula we see that the denominator (seen by restriction as being in the Grothendieck group of T) is equal to the following expression, Ω_M being the Weyl group of M:

$$\frac{\sum_{w\in\Omega}\varepsilon(w)w\rho}{\sum_{w\in\Omega_M}\varepsilon(w)w\rho} = \rho\rho_M^{-1}\prod_{\substack{\alpha\in\Delta_+^G\\ \alpha\notin\Delta_+^M}}(1-\alpha). \tag{2.5}$$

Since the centralizer of x is exactly M, the right-hand side of (2.5) is non-zero at x.

Now consider again (2.4). The denominator (at x) is fixed and non-zero when λ varies. We must show that for $r\to+\infty$ and $\lambda=\lambda_r$ each term

$$\varepsilon(w)\,\mathrm{ch}_M(V^M_{w(\lambda+\rho)-\rho}), \tag{2.6}$$

divided by $\dim(V_r)$, tends to 0. Since x is central in M, (2.6) is essentially the dimension of the module. By Weyl's dimension formula, we must consider the behaviour of

$$\frac{\prod_\beta\langle\beta, w(\lambda+\rho)-\rho+\rho_M\rangle}{\prod_\alpha\langle\alpha,\lambda+\rho\rangle}. \tag{2.7}$$

Here α ranges over positive roots of (G,T) and β over positive roots of (M,T). Now (2.7) is

$$\frac{\prod_\gamma\langle\gamma,\lambda+\rho-w(\rho-\rho_M)\rangle}{\prod_\alpha\langle\alpha,\lambda+\rho\rangle},$$

where now γ ranges over $w^{-1}\Delta_M^+\subset\Delta_G^+$, a subset of the α's. If α is equal to γ, the corresponding ratio is constant (if α vanishes on the direction of the λ's) or tends to 1. (Note that all terms are positive.) So we only have to show:

LEMMA 2.4: *Fix $w\in\Omega^M$. There exists $\alpha\notin w^{-1}\Delta_M^+(\alpha\in\Delta_G^+)$ such that $\langle\alpha,\lambda_1\rangle\neq 0$.*

Indeed, let Δ_{M_1} be the subset of $\alpha\in\Delta_G$ such that $\langle\alpha,\lambda_1\rangle=0$ (it is the root system of a Levi subgroup of the complexification of G). If the Lemma is false, $\Delta_G^+\subset w^{-1}\Delta_M^+\cup\Delta_{M_1}$. In particular, $\Delta_G\subset w^{-1}\Delta_M\cup\Delta_{M_1}$. This says that the root system of G is the union of two roots systems associated to proper Levi subgroups.

Thus the following lemma completes the proof:

LEMMA 2.5: *Assume Δ is an irreducible root system in a vector space V of dimension at least 2. Then Δ is not contained in the union of two hyperplanes $(H;H')$ of V.*

The proof was indicated to us by J. Y. Hée. Let Π be a basis of Δ and Γ the Dynkin diagram associated to Π. Assume $\Delta\subset H\cup H'$. Since Π spans V, there exists $\alpha \in \Pi$ such that $\alpha \notin H'$ and $\alpha' \in \Pi$ such that $\alpha' \notin H$. Then $\alpha \in H$, $\alpha' \in H'$. We can choose (α, α') such that their distance in Γ is minimal. Let $(\alpha_0 = \alpha, \alpha_1, \dots, \alpha_r = \alpha')$ be the path joining α and α' in Γ. Since the distance of α, α' is minimal, $\alpha_i \in H \cap H'$ for $i = 1, \dots r-1$. Set $\beta = \alpha_0 + \alpha_1 + \cdots + \alpha_r$. Then $\beta \in \Delta$ (since $(\alpha_0 + \cdots + \alpha_s, \alpha_{s+1}) < 0$ for $1 \leq s \leq r-1$) so $\beta \in H \cup H'$. However, if $\beta \in H'$, $\alpha = \beta - \alpha_1 \cdots - \alpha_r \in H'$; if $\beta \in H$, $\alpha' \notin H$, contradiction.

Remark: The proof remains correct in essence, in the case of the system of **real** roots in a Kac–Moody algebra associated to an indecomposable generalized Cartan matrix.

In conclusion, we note that this proof of Theorem 2 extends to the action of a symmetric Hecke operator T_S, defined by a compact connected Lie group G and acting on $L^2(G)$: since we want to produce a lower bound it suffices to consider $L^2(G/H)$ where H is a subgroup of G such that (G, H) is symmetric. If G/H has rank one, essentially the same method works; however, the centralizers will not always be Levi subgroups and one needs an extension of Kostant's formula to these centralizers.

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